MOTION OF STABILIZED GYROSCOPIC SYSTEMS ON A MOVING BASE

(DVIZHENIYE STABILIZIROVANNYKH GIROSKOPICHESKIKH SISTEM NA PODVIZHNOM OSNOVANII)

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The paper gives a justification for a well-known method in the applied theory of gyroscopes [1-6].

1. Consider the general case of motion of a stabilized gyroscopic system, assuming an arbitrary dependence on time of the motion of the base and of the mass of the gyroscopic system. Also the proper rotations [spin] of the gyroscopes are assumed to be nonstationary.

The kinetic energy of a gyroscopic system with r gyroscopes and s positional coordinates q_i has the form

$$T = T_2' + T_1' + T_0' + \frac{1}{2}\sum_{k=1}^{r} C_k\left(\dot{\varphi}_k + \sum_{j=1}^{s} a_j^{\,k}\dot{q}_j + a_0^{\,k}\right)^2 \qquad (1.1)$$

Here T_2', T_1' and T_0' denote the quadratic, linear and zero form in terms of positional velocities $\dot{q}_i$ respectively; ϕ_k are cyclic coordinates, denoting the angles of proper rotations of the gyroscopes, C_k is the axial moment of inertia of the kth gyroscope, $a_j^{\,k}$ is the cosine of the angle between the angular velocity vector $\dot{q}_j$ and the axis of the kth gyroscope, $a_0^{\,k}$ is the projection of the angular velocity of the base on the axis of the kth gyroscope.

The coefficients $a_j^{\,k}$ and $a_0^{\,k}$ depend on the positional coordinates and time.

Let the generalized usual and reaction forces [7], corresponding to the cyclic coordinates, be explicit functions of time. The Lagrange equations of the second kind for the cyclic coordinates ϕ_k are

$$\dot{\varphi}_k + \sum_{j=1}^{s} a_j^{\,k}\dot{q}_j + a_0^{\,k} = H + h_k \qquad (1.2)$$

where H, being a constant, will be considered sufficiently large, $H > h_k$

and h_k are functions of time containing $r-1$ constants.

Introducing the Routh function

$$R(\dot{q}_i,\ q_i,t) = T^* - \sum_{k=1}^{r} C_k(H+h_k)\ \dot{\varphi}_k, \qquad T^* = T(\dot{q}_i q_i,\ \dot{\varphi}_k(\dot{q}_i,\ q_i, H, t))$$

the Lagrange equations of the second kind for the positional coordinates assume the form

$$\frac{Д}{Дt}\frac{DR}{D\dot{q}_i} - \frac{DR}{Dq_i} = Q_i + \Psi_i \tag{1.3}$$

The notations used for the derivatives are those from the mechanics of variable mass [7], the masses being considered as fixed when calculating these derivatives. Ψ_i denote generalized reaction forces.

By virtue of (1.1) and (1.2) the Routh function has the form

$$R = T_2' + T_1' + T_0' + \sum_{k=1}^{r} C_k(H+h_k)\left(\sum_{j=1}^{s} a_j^{\ k}\dot{q}_j + a_0^{\ k}\right) - \frac{1}{2}\sum_{k=1}^{r} C_k(H+h_k)^2 \tag{1.4}$$

where the last term is an explicit function of time which, when writing down the equations (1.3), can be omitted. For a stabilized gyroscopic system it is in order to consider the linearized equations.

Therefore, restricting ourselves in the equations (1.3) to small terms of the first order with respect to q_i and $\dot{q}_i$, we obtain equations of the form

$$a_{ij}\ddot{q}_j + (b_{ij} + Hg_{ij})\,\dot{q}_j + (c_{ij} + Hd_{ij})q_j = f_i + HF_i \tag{1.5}$$

Here $\| a_{ij} \|$ is the matrix of a positive definite quadratic form, the coefficients in equations (1.5) are explicit functions of time which do not depend on the parameter H.

Here and below the presence of repeated indices in the factors denotes summation. For stabilized gryoscopic systems the proper motions of the system, determined by the general solution of the corresponding homogeneous system of equations (1.5), are bounded and cannot be of higher order than zero with respect to H.

Investigating gyroscopic systems [1-6], usually the method of applied theory of gyroscopes is used. The equations of motion obtained by this method do not take into account the kinetic moments of the elements of suspension of the gyroscopic system nor those of the gimbals of the gyroscopes, nor the equatorial components of the kinetic moments of the rotors themselves, nor the kinetic moments of the motors. Therefore, according to the method of applied theory of gyroscopes the kinetic energy and the Routh function have respectively the forms (1.1) and (1.4) in which

$$T_2' = T_1' = T_0' = 0$$

Equations (1.2) and the generalized forces $Q_i + \Psi_i$ have the same form.

Equations (1.5) can be written in the form

$$(b_{ij}^* + Hg_{ij})\, \dot{g}_j + (c_{ij}^* + Hd_{ij})\, g_j = f_i^* + HF_i \tag{1.6}$$

Here the asterisk denotes that the indicated coefficients differ from those used earlier. For convenience the positional coordinates in this case are denoted by q_j.

Remarks. (a) In real gyroscopic systems the coefficients d_{ij} and F_i can be of the order of the angular velocity of Barth's rotation, the latter being a sufficiently small quantity, while the coefficients c_{ij} can be of the order of the pendulum momentum which for certain gyroscopic devices is close to the quantity H. Therefore, in the case of a concrete gyroscopic system its particular properties must be taken into account.

(b) If the generalized usual and reaction forces are not explicit functions of time, we arrive at the equations of the form (1.5) and (1.6), if from the consideration of the Lagrange equations of the second kind for the cyclic coordinates, we can determine the functions $\phi_k = \phi_k(q_i, q_i, t)$.

2. Assume that the determinant of the matrix of the gyroscopic terms $\| g_{ij} \|$ is different from zero and consider the solutions of the equations (1.6) corresponding to arbitrary initial conditions $g_i^0 = g_i^0$. The variables g_i are of the order of the initial values, i.e. of the order zero with respect to H. The order of q_i is not greater than zero. Let g_i be of a certain order O' with respect to H. Consider the solution of equations

$$(b_{ij} + H\dot{g}_{ij})\, \dot{q}_{j1} + (c_{ij} + Hd_{ij})\, q_{j1} = f_i + HF_i \tag{2.1}$$

corresponding to the initial conditions $g_{j1}^0 = g_j^0$.

By virtue of equations (1.6) and (2.1) we obtain for the variables $x_i = q_{i1} - g_i$ the equations

$$(H^{-1}b_{ij} + g_{ij})\, \dot{x}_j + (H^{-1}c_{ij} + d_{ij})\, x_j = H^{-1}\left[(b_{ij}^* - b_{ij})\, \dot{g}_j + (c_{ij}^* - c_{ij})\, g_j + f_i - f_i^*\right]$$

from which follows

$$\{q_{i1} - g_i,\ \dot{q}_{i1} - \dot{q}_i,\ \ddot{q}_{i1} - \ddot{g}_i\} = O(H^{-1}) \tag{2.2}$$

Denote by $q_i^{(1)}$ the motion of the gyroscopic system (1.5) corresponding to the initial conditions

$$q_j^{(1)\circ} = q_j^\circ, \ [b_{ij}(0) + Hg_{ij}(0)]\dot{q}_j^{(1)\circ} + [c_{ij}(0) + Hd_{ij}(0)]\, q_j^\circ = f_i(0) + HF_i(0) \quad (2.3)$$

By virtue of formulas (1.5), (2.1) and (2.3) the variables $y_i = q_i^{(1)} - q_{i1}$ are particular solutions, with zero initial values, of the following system of equations

$$a_{ij}\ddot{y}_j + (b_{ij} + Hg_{ij})\,\dot{y}_j + (c_{ij} + Hd_{ij})\, y_j = -\sum_{j=1}^{s} a_{ij}\ddot{q}_{j1} \quad (2.4)$$

From formulas (2.2) and (2.4) it follows that the variables $\dot{y}_j$ and y_j are of the order O'', O'' being the largest of the quantities O' and H^{-1}.

The solution of equations (1.5), with arbitrary initial values q_i^0, $\dot{q}_i^0$, is equal to $q_i = q_i^{(1)} + q_i^{(2)}$, where $q_i^{(2)}$ is the solution of the corresponding homogeneous equations of system (1.5), having the initial values

$$q_i^{(2)\circ} = 0, \qquad \dot{q}_i^{(2)\circ} = \dot{q}_i^\circ - \dot{q}_i^{(1)\circ} \quad (2.5)$$

Putting $\tau = Ht$ and denoting by primes the derivatives with respect to τ, we obtain

$$a_{ij}q_j^{(2)''} + (H^{-1}b_{ij} + g_{ij})\, q_j^{(2)'} + H^{-2}(c_{ij} + Hd_{ij})\, q_j^{(2)} = 0$$

Consider the solution corresponding to the initial values (2.5) of the following system of equations

$$a_{ij}q_{j2}'' + (H^{-1}b_{ij} + g_{ij})\, q_{j2}' = 0 \quad (2.6)$$

The variables q_{j2}' have the same order with respect to H as the initial values, i.e. H^{-1}. The variables $z_j = q_j^{(2)} - q_{j2}$ are particular solutions, with zero initial values, of the following equations

$$a_{ij}z_j'' + (H^{-1}b_{ij} + g_{ij})\, z_j' + H^{-2}(c_{ij} + Hd_{ij})\, z_j = -H^{-2}(c_{ij} + Hd_{ij})\, q_{j2} \quad (2.7)$$

Solving equations (2.6) with respect to q_{j2}' and integrating with respect to τ, we obtain [8] that the order of q_{j2} is equal to H^{-1}. Computing the particular solution of equations (2.7) corresponding to zero initial values, the integration with respect to τ is to be extended from 0 to Ht. Therefore we obtain $\{z_j', z_j\} = o(H^{-1})$.

Consequently we obtain the following estimates

$$q_i = g_i + o'', \qquad \dot{q}_i = \dot{g}_i + \dot{q}_i^{(2)} + O''$$

which serve as the basis for the applied theory of gyroscopes. It should be emphasized that $\dot{z}_j$ is of the order zero with respect to H and in the case under consideration $\dot{q}_i^{(2)}$ cannot be replaced by the velocities $\dot{q}_{i2}$.

Also it cannot be asserted that the order of q_{i1} is equal to H^{-1} as is the case for gyroscopic systems on a fixed base [8].

The result obtained above can be immediately extended to the case of an automatically regulated gyroscopic system with k additional equations

$$\begin{gathered} a_{ij}\ddot{q}_j + (b_{ij} + Hg_{ij})\dot{q}_j + (c_{ij} + Hd_{ij})q_j + e_{i\nu}\dot{q}_\nu + f_{i\nu}q_\nu = f_i + HF_i \\ A_{\mu\nu}\ddot{q}_\nu + B_{\mu\nu}\dot{q}_\nu + C_{\mu\nu}q_\nu + D_{\mu j}\dot{q}_j + E_{\mu j}q_j = f_\mu + HF_\mu \end{gathered} \tag{2.8}$$

Here the indices μ and ν vary from $s + 1$ to $s + k$ and correspond to the additional equations of automatic damping. Also in this case the passage to simplified equations, obtained by dropping the terms $a_{ij}\ddot{q}_j$ in the first s equations of system (2.8), leads to equations in which terms of the order equal to the largest of H^{-1} and of the order of the accelerations $\ddot{q}_j$ in the simplified equations, are not taken into account.

3. As an illustrative example consider Sperry's ship gyrocompass "Mark II" [3]. If we assume that the ship moves with a constant velocity and neglect the eastward component of the velocity when compared with the circumferential velocity of Earth's rotation which on the equator assumes the value of 1670 km/hr, then the equations of motion have the form [3]

$$\begin{gathered} A\ddot{\alpha} + H\dot{\beta} + H\Omega\cos\varphi\alpha = H\frac{v_N}{R} \\ B\ddot{\beta} - H\dot{\alpha} + (lP + H\Omega\cos\varphi)\beta = H\Omega\sin\varphi \end{gathered} \tag{3.1}$$

where α and β are the angles of inclination of the axis of the gyrocompass, A and B are the moments of inertia, lP the pendulum moment, v_N the projection of the velocity of the ship along the direction "north", R = 6370 km is the radius of the Earth, $\Omega = 7.29 \times 10^{-5}\text{sec}^{-1}$ is the angular velocity of the Earth's rotation and $H = 2.76 \times 10^9 \text{g cm}^2/\text{sec}$.

The forced solution of system (3.1) has the form

$$\alpha^* = \frac{v_N}{R\Omega\cos\varphi}, \qquad \beta^* = \frac{H\Omega\sin\varphi}{H\Omega\cos\varphi + lP}$$

In order to have β^* small, the pendulum moment lP must be selected much larger than $H\Omega$. For the Sperry compass $lP = 7.23 \times 10^7$ dyn cm^2, i.e. the order is close to H, further $H\Omega = 2.01 \times 10^5$ g cm^2/sec^2.

Dropping the second order derivatives in system (3.1) we arrive at equations of the form (2.1)

$$-H\dot{\alpha}_1 + (lP + H\Omega\cos\varphi)\beta_1 = H\Omega\sin\varphi, \qquad H\dot{\beta}_1 + H\Omega\cos\varphi\alpha_1 = H\frac{v_N}{R} \tag{3.2}$$

In deriving equations (3.1), the kinetic moments of the elements of suspension and of the gimbals of the gyroscopes and also the kinetic moments of the motors have not been taken into account. Therefore, in the case under consideration equations (1.6) have the form (3.2).

Differentiating equations (3.2) we obtain

$$\ddot{\alpha}_1 = H^{-1}(lP + H\Omega\cos\varphi)\left(\frac{v_N}{R} - \Omega\alpha_1\cos\varphi\right)$$
$$\ddot{\beta}_1 = \Omega^2\sin\varphi\cos\varphi - H^{-1}\Omega\cos\varphi\,(lP + H\Omega\cos\varphi)\,\beta_1 \tag{3.3}$$

Since

$$v_N R^{-1} < \Omega < lPH^{-1},$$

the order O' for the Sperry compass is equal to the order of $lP\Omega H^{-1}$. We have $lP\Omega H^{-1} = 2 \times 10^{-6}\text{sec}^2$. Therefore, the order of O' is close to $H^{-1/2}$, and to within terms of this order, the variables α_1 and β_1 replace α and β.

As an example for the additional equation of automatic damping we can take the equation of flow for the damping fluid in the gyrocompass of Anschutz [3]

$$\dot{\theta} = -F(\beta + \theta)$$

where $F =$ const is a factor of flow and θ is the angle of inclination of the damping fluid. If we neglect the term $H\Omega\cos\phi$ as compared with lP, then the simplified equations assume the form of the well-known equations of Geckeler-Krylov for the uniform motion of the ship

$$-H\dot{\alpha}_1 + lP\beta_1 + c\theta_1 = H\Omega\sin\varphi,$$
$$\dot{\beta}_1 + \Omega\cos\varphi\alpha_1 = \frac{v_N}{R}, \qquad \dot{\theta}_1 = -F(\beta_1 + \dot{\theta}_1)$$

Hence, the motion obtained from these equations determines the actual motion of the gyroscopic system to within terms of the order equal to the largest of the quantities H^{-1}, $lP\Omega H^{-1}$ and $H\Omega l^{-1}P^{-1}\cos\phi$.

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